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On the skewness of sea-surface elevation

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Surface skewness is a statistical measure of the vertical asymmetry of the air–sea interface – exemplified by the sharp crests and rounded troughs of surface gravity waves. Some authors have proposed a constant ratio between surface skewness and the 'significant slope' of the waves. Here it is shown theoretically that no such simple relation is to be expected.

Wave records are of at least two different types; Eulerian (as made with a fixed probe) or Lagrangian (as with a free-floating buoy). The corresponding statistical properties are examined. At first sight it might appear that the degree of skewness in corresponding records would be different. However it is shown that to lowest order the skewness is invariant; only the apparent mean level is different, at second order.

1. Introduction

With the advent of radar altimetry from orbiting satellites, and its application to the measurement of ocean waves, currents and surface winds, certain questions concerning the statistical properties of surface waves have come increasingly to the fore. Among these is the magnitude of the surface 'skewness', defined as follows. If we suppose the vertical displacement ζ of the ocean surface to be recorded as a function of the time t at some fixed location, then, in a given sea state, the elevation ζ will have a probability density $p(\zeta)$, say. For waves of small slopes, $p(\zeta)$ is known to be nearly Gaussian (see for example Longuet-Higgins 1957). However, in steep waves, including sometimes very short gravity or capillary waves, $p(\zeta)$ becomes asymmetric about its mean level $\zeta = 0$ and may have an appreciable skewness λ_3, as defined in terms of the second and third cumulants of $p(\zeta)$. One familiar manifestation of surface skewness is the up–down asymmetry of a steep gravity wave, in which the crests are more peaked, the troughs flatter or more rounded.†

The value of λ_3 can be related to the nonlinear dynamics of free surface waves. Phillips (1961) first showed theoretically that λ_3 was of the same order of magnitude as the r.m.s. surface slope. Longuet-Higgins (1963) gave a detailed theory, deriving the skewness and kurtosis of $p(\zeta)$ in terms of the underlying frequency spectrum of ζ. However, in some more recent papers (Walsh 1979; Huang & Long 1980; Huang *et al.* 1981) there have been suggestions, made on empirical grounds, that there exists a simple linear relationship between λ_3 and a quantity s, the 'significant slope', defined in terms of the frequency spectrum of ζ. Thus Huang & Long (1980) proposed that

$$\lambda_3 = 8\pi s. \tag{1.1}$$

† This type of asymmetry is to be distinguished from the horizontal asymmetry in some wind waves, which is related to the distribution of the surface slopes (see Longuet-Higgins 1982).

Such a relation would indeed be convenient. However, one of the conclusions of the present paper is that no such simple relation exists.

In the first part of the paper, which is theoretical, we introduce a simple model of the wavefield, appropriate to long-crested waves with a narrow frequency spectrum. In this case it is easy to derive a simple relation between the skewness and the significant slope. The result (2.16) is shown in §3 to be consistent with the more general theory of Longuet-Higgins (1963) after correction of an elusive factor. The more general theory is then used to investigate the effects of finite spectral bandwidth and varying shape of the frequency spectrum, on the ratio between λ_3 and s. The relation is found not to be unique. In §4 we review recent observations of λ_3 in the light of our theoretical results.

In situ measurements of waves are often made with different types of instrument, giving rise to wave records of either Eulerian or Lagrangian type. The latter, for example, would include measurements with a free-floating buoy. Are there any differences in the skewness as evidenced by different types of measurement? This question is investigated in §§5 and 6. In §5 we obtain a general relation, correct to second order, between the two types of measurement ((5.9)) and apply it to the narrow-band spectral model. In §6 we consider a more general case. The conclusions are summarized in §7.

2. Model for a narrow spectrum

Suppose first that the waves are long-crested and have a narrow frequency spectrum, in the sense of Longuet-Higgins (1957). Choosing the horizontal x-axis in the direction of propagation we may write

$$\zeta(x,t) = a\cos\theta + \tfrac{1}{2}a^2 k\cos 2\theta + O(a^3k^2), \tag{2.1}$$

where a represents the local wave amplitude, k is a fixed wavenumber and θ is the phase function

$$\theta = kx - \sigma t + \epsilon. \tag{2.2}$$

Here σ is the (fixed) radian frequency and a and ϵ vary slowly with x and t. The first term on the right of (2.1) represents a linear, sinusoidal wave, of slowly varying amplitude and phase. The second term represents the nonlinear correction appropriate to a deep-water gravity wave of uniform amplitude (see for example Lamb 1932, p. 417).

By linear theory, and for a narrow spectrum, the distribution of wave heights $2a$ is Rayleigh:

$$p(a) = \frac{2a}{\bar{a}^2}\,\mathrm{e}^{-a^2/\bar{a}^2}, \tag{2.3}$$

where $\bar{a}$ is the r.m.s. value of a. We note that, even after the addition of the second, nonlinear term on the right of (2.1), the crest-to-trough wave height is still equal to $2a$, if we neglect quantities of order a^3k^2. We shall assume that a is distributed according to (2.3) in the nonlinear case also, and that the phase θ is distributed uniformly in $(0, 2\pi)$, as in narrowband linear theory (see Longuet-Higgins 1963). Thus the joint density of a and θ is

$$p(a,\theta) = \frac{a}{\pi\bar{a}^2}\,\mathrm{e}^{-a^2/\bar{a}^2}. \tag{2.4}$$

From (2.1) and (2.4) we may at once calculate the surface skewness. The rth moments μ_r being defined by

$$\mu_r = \int_0^\infty \int_0^{2\pi} \zeta^r p(a, \theta)\, \mathrm{d}a\, \mathrm{d}\theta, \tag{2.5}$$

we easily find $\mu_1 = 0$ and

$$\mu_2 = \tfrac{1}{2}\bar{a}^2, \quad \mu_3 = \tfrac{3}{4}\bar{a}^4 k, \quad \mu_4 = \tfrac{3}{4}\bar{a}^4 k^2. \tag{2.6}$$

Hence the cumulants κ_r are given by $\kappa_1 = 0$ and

$$\kappa_2 = \tfrac{1}{2}\bar{a}^2, \quad \kappa_3 = \tfrac{3}{4}\bar{a}^4 k, \quad \kappa_4 = 0, \tag{2.7}$$

to the present approximation. The coefficient of skewness is then

$$\lambda_3 \equiv \frac{\kappa_3}{\kappa_2^{\frac{3}{2}}} = \frac{3}{\surd 2}\bar{a}k \tag{2.8}$$

and the coefficient of kurtosis is

$$\lambda_4 \equiv \frac{\kappa_4}{\kappa_2^2} = 0, \tag{2.9}$$

to this order.

These results should agree with the expressions for the cumulants given by Longuet-Higgins (1963) for a general long-crested frequency spectrum $F(\sigma)$. These are (after correction† of a factor $\frac{1}{2}$ in his equation (3.7))

$$\kappa_1 = 0, \tag{2.10}$$

$$\kappa_2 = \int_0^\infty F(\sigma)\, \mathrm{d}\sigma, \tag{2.11}$$

$$\kappa_3 = 3\int_0^\infty \int_0^\infty \min(k, k')F(\sigma)\; F(\sigma')\, \mathrm{d}\sigma\, \mathrm{d}\sigma'. \tag{2.12}$$

κ_4 was of higher order, as noted. If in (2.11) and (2.12) we introduce the narrow spectrum

$$F(\sigma) = \tfrac{1}{2}\bar{a}^2\, \delta(\sigma - \sigma_0), \tag{2.13}$$

where δ denotes the Dirac delta function, we retrieve precisely (2.8).

Consider now the relation of the skewness to the 'significant slope' s. This was defined by Huang & Long (1980) as

$$s = \bar{\zeta}/L_{\mathrm{p}} \tag{2.14}$$

where a bar denotes the r.m.s. value and L_{p} is the wavelength corresponding to the peak in the spectrum. So in our model

$$s = \frac{\kappa_2^{\frac{1}{2}} k}{2\pi} = \frac{\bar{a}k}{2^{\frac{3}{2}}\pi} \tag{2.15}$$

and from (2.8) we have the relation

$$\lambda_3 = 6\pi s. \tag{2.16}$$

† See Bitner (1976) and Bitner-Gregersen (1980). There should be a factor $\frac{1}{2}$ multiplying the right-hand sides of equations (3.7), (3.9), (3.12) and (3.17). Hence the numerical factors in (3.14), (3.15) and (3.16) should be 3, 6 and 6 respectively. The ratio λ_3/L in (3.24), (3.25) and table 1 is unaffected.

Lastly, in this section, we note that by (2.1) the r.m.s. surface slope $\bar{\zeta}_x$ is given by

$$\bar{\zeta}_x^2 = \tfrac{1}{2}(\bar{a}k)^2, \tag{2.17}$$

so that from (2.7)

$$\lambda_3 = 3\bar{\zeta}_x. \tag{2.18}$$

This can also be written

$$\langle\zeta^3\rangle = 3\langle\zeta_x^2\rangle^{\frac{1}{2}}\langle\zeta^2\rangle^{\frac{3}{2}}. \tag{2.19}$$

This is the correct form† of a relation first given by Phillips (1961, p. 154) in which the factor on the right was given as $\frac{3}{8}$.

3. Effects of finite bandwidth

We now generalize some of the results of §2 to seas that are still long-crested, that is to say unidirectional, but have a non-zero bandwidth. For such waves, (2.10)–(2.12) will apply, and from the definitions of λ_3 and s given above we have

$$\lambda_3 = 12\pi s \frac{I_2}{I_1^2}, \tag{3.1}$$

where

$$I_1 = \sigma_\mathrm{p} \int_0^\infty F(\sigma)\,\mathrm{d}\sigma \tag{3.2}$$

and

$$I_2 = \int_0^\infty \left\{ \int_0^{\sigma'} \sigma^2 F(\sigma)\,\mathrm{d}\sigma \right\} F(\sigma')\,\mathrm{d}\sigma'. \tag{3.3}$$

In (3.3) we used the dispersion relation $\sigma^2 = gk$, and in (3.2) the relation $\sigma_\mathrm{p}^2 = 2\pi g/L_\mathrm{p}$ for the radian frequency σ_p at the spectral peak.

We shall now evaluate the factor I_2/I_1^2 on the right of (3.1) for some typical wave spectra.

Consider first spectra of the special form

$$F(\sigma) = \begin{Bmatrix} \alpha\sigma^{-n}, & \sigma > \sigma_\mathrm{p} \\ 0, & \sigma < \sigma_\mathrm{p} \end{Bmatrix}, \tag{3.4}$$

n being a constant greater than 3. (The case $n = 5$ corresponds to the Phillips spectrum, Mark I). Substituting into (3.1) we find

$$\lambda_3 = \frac{6(n-1)}{n-2}\pi s. \tag{3.5}$$

This is the correct version of equation (5.9) of Huang *et al.* (1983) and the result, for $n = 5$, given previously by Jackson (1979). When $n \to \infty$, (3.5) reduces to (2.16), as would be expected. When $n = 5$ (and only in this case) (3.5) agrees with the empirical relation (1.1) given by Huang & Long (1980).

To assess roughly the dependence of λ_3/s upon the spectral width we may introduce the spectral-width parameter ν defined by

$$\nu^2 = \frac{m_0 m_2}{m_1^2} - 1 \tag{3.6}$$

(cf. Longuet-Higgins 1980), where m_r denotes the rth spectral moment

$$m_r = \int_0^\infty \sigma^r F(\sigma)\,\mathrm{d}\sigma. \tag{3.7}$$

† We are indebted to Professor Phillips for verifying this statement.

n	ν	$\lambda_3/\pi s$
4	0.5774	9
5	0.3536	8
6	0.2582	7.5
10	0.1260	6.75
100	0.0102	6.06
∞	0	6

TABLE 1. Dimensionless parameters ν and $\lambda_3/\pi s$ for the ideal spectrum (3.4)

(Note that $m_0 = \mu_2$). From (3.4), (3.6) and (3.7) we find

$$\nu^2 = \frac{1}{(n-1)(n-3)}. \tag{3.8}$$

Table 1 gives the results for some integer values of n. It suggests that, as the spectral width decreases, so also does the ratio λ_3/s.

Next consider a generalized form of the Pierson–Moskowitz (P–M) spectrum used successfully by Liu (1983, 1985):

$$F(\sigma) = \alpha\sigma^{-n}\,\mathrm{e}^{-(\beta/\sigma)^m}, \tag{3.9}$$

which has a peak at $\sigma = \sigma_\mathrm{p} = \beta(m/n)^{1/m}$. When $m = 4$ and $n = 5$ (3.9) gives the well-known P–M spectrum, while for $m = 4$ and n arbitrary we obtain the Wallops spectrum (Huang *et al.* 1981). Lastly when $n = 5$ and m is arbitrary we obtain the spectrum used by Longuet-Higgins (1980).

From (3.7) we find, when $r < (n-1)$,

$$m_r = \frac{\alpha}{m\beta^{n-r-1}}\,\Gamma\left(\frac{n-r-1}{m}\right) \tag{3.10}$$

and so from (3.8)

$$\nu^2 = \Gamma\left(\frac{n-3}{m}\right)\Gamma\left(\frac{n-1}{m}\right)\Big/\Gamma\left(\frac{n-2}{m}\right)^2 - 1. \tag{3.11}$$

Furthermore from (3.1) we find (see Appendix) that

$$\lambda_3 = 3\pi s\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{n}{m}\right)^{2/m} {}_2F_1\left(1, \frac{2(n-2)}{m}; \frac{n-1}{m}+1; \tfrac{1}{2}\right)\Big/\Gamma\left(\frac{n-1}{m}\right)^2 \tag{3.12}$$

where ${}_2F_1$ is a generalized hypergeometric function, from which numerical values may readily be computed.

Results for various values of n and m are given in table 2. It can be clearly seen that the ratio of skewness to significant slope varies widely, is crucially dependent on the form of the wave spectrum and is not simply a function of the bandwidth parameter ν.

We note that (3.1) and (3.3) apply only to long-crested waves. However, for a more general three-dimensional spectrum it has been shown that the coefficient of skewness λ_3 satisfies

$$0.44L_3 \leqslant \lambda_3 \leqslant 1.01L_3, \tag{3.13}$$

where L_3 denotes the corresponding skewness for a long-crested sea (Longuet-Higgins 1963, pp. 469–470). The proof of this result is unaffected by the presence of a factor $\frac{1}{2}$ in equation (3.7)*.†

† We use a star * to denote equation numbers in Longuet-Higgins (1963).

$n \backslash m$	ν 4	5	6	7	$\lambda_3/\pi s$ 4	5	6	7
4	0.64	0.62	0.61	0.60	3.66	2.16	1.41	0.98
4	0.42	0.41	0.39	0.39	6.96	4.33	2.88	2.03
6	0.33	0.31	0.30	0.29	10.26	6.95	4.81	3.46
7	0.28	0.26	0.25	0.24	12.66	9.51	6.97	5.17

TABLE 2. Values of ν and $\lambda_3/\pi s$ for the generalized P–M spectrum (3.9)

4. Discussion

Consider first the observations reported by Huang & Long (1980). In their figure 6 (where λ_3 is plotted against s) the data fall mainly into two groups: the field observations, for which $0 < s < 0.02$, and the laboratory data, for which $0.02 < s < 0.04$. The former show considerably more scatter. Thus their empirical result (1.1) is probably weighted in favour of the laboratory measurements. The field data alone would not suggest such a definite relationship.

We note that a similar scatter in field data is reported by McClain, Chen & Hart (1982, figure 3). This certainly supports our conclusion that the relation (1.1) is not unique.

There may also be systematic differences between field and laboratory data arising from different ranges of the parameter u^*/c (where u^* is the wind friction velocity and c the phase speed of the dominant waves). For waves in the open ocean u^*/c is typically of order 0.1, compared with values of order 1 for wind waves in the laboratory (Phillips 1977, p. 129). The laboratory data of Huang & Long (1980) include results with $u^*/c > 0.6$; below this value their measurements of skewness are considerably more scattered.

A non-Gaussian model of the sea surface somewhat similar to that in §2 above was proposed by Huang *et al.* (1983), except that they include a term $\frac{1}{2}a^2k$ in (2.1), as well as higher-order terms. Such a term, however, does not arise dynamically in deep water; it would correspond to a local change in the mean surface level, i.e. a 'wave set-up'. Although such terms are significant in shallow water (Longuet-Higgins & Stewart 1962, 1964), nevertheless in deep water they become negligible when $\Delta k \ll k$, where Δk is the spectral bandwidth, that is when the spectrum is narrow and there are many waves in a group – the situation considered in §2.

5. Lagrangian measurements: narrow spectrum

In determining the skewness of the surface elevation from instrumental records, some attention must be paid to the method of measurement, since different methods may give apparently different answers.

The definition of surface skewness given in §2 applies directly to measurements made with a fixed probe or wave staff. However an alternative method of observation is often used, in which the vertical displacement is derived by twice integrating the vertical acceleration in a free, or almost free, floating buoy. To first order in ak the two wave records are equivalent, but to second order, which is required for an assessment of the skewness, the records are different, as we shall show.

Note first that an irrotational deep-water Stokes wave can be considered as the

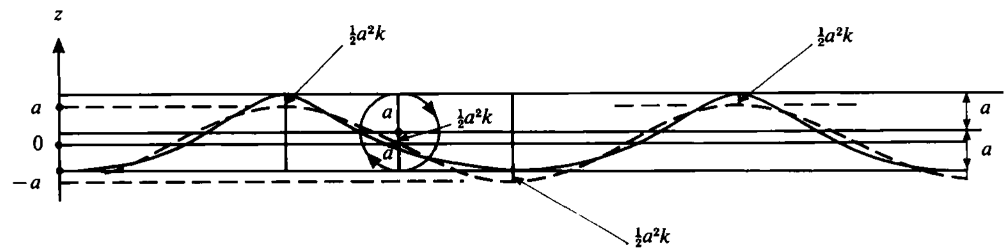

FIGURE 1. Sketch of orbital motion in a steep, irrotational wave, when the frame of reference moves with the Stokes drift velocity for surface particles. The broken curve corresponds to linear theory.

sum of a rotational Gerstner wave (Lamb 1932, section 251) in which the particles describe perfectly circular orbits, plus a steady, second-order Stokes drift. The superposition of the two motions is accurate to third order in the wave steepness (see Dubreil-Jacotin 1934). Hence in a Stokes wave each particle at the surface describes a circular path, if seen in a frame of reference moving with the steady drift; see figure 1. Moreover, its orbital velocity in this circular path is constant (see Lamb 1932). So apparently there is no asymmetry in its vertical displacement, to third order, and one might expect the corresponding skewness in the wave record to be small.

To analyse the situation further, let z be the vertical coordinate and u, w the horizontal and vertical components of the velocity. We shall suppose that

$$\zeta = \gamma\zeta^{(1)} + \gamma^2\zeta^{(2)} + \dots, \tag{5.1}$$

where γ is a small parameter proportional to the maximum surface slope, and we shall use suffices L and E to denote quantities following a particle or with fixed spatial coordinates, respectively. Then the horizontal displacement of a particle is given by

$$\Delta x = \int u_{\mathrm{L}}\,\mathrm{d}t = \int u_{\mathrm{E}}\,(\boldsymbol{x}+\Delta\boldsymbol{x})\,\mathrm{d}t \tag{5.2}$$

and on expanding in a Taylor series about $\boldsymbol{x}$, we find

$$\Delta x = \int u_{\mathrm{E}}(\boldsymbol{x})\,\mathrm{d}t + \Delta\boldsymbol{x}\cdot\int \boldsymbol{\nabla u}_{\mathrm{E}}(\boldsymbol{x})\,\mathrm{d}t + \dots. \tag{5.3}$$

Thus to first order in γ we have simply

$$\Delta x = \int u\,\mathrm{d}t. \tag{5.4}$$

In a similar way the kinematic surface condition leads to

$$\frac{\partial\zeta}{\partial t} = w \tag{5.5}$$

correct to first order, and

$$\zeta_{\mathrm{L}} = \zeta_{\mathrm{E}}(x+\Delta x) = \zeta_{\mathrm{E}} + \Delta x\,\frac{\partial\zeta}{\partial x} \tag{5.6}$$

correct to order γ^2. But, to first order,

$$\frac{\partial\zeta}{\partial x} = \frac{\partial}{\partial x}\int\frac{\partial\zeta}{\partial t}\,\mathrm{d}t = \frac{\partial}{\partial x}\int w\,\mathrm{d}t = \int\frac{\partial w}{\partial x}\,\mathrm{d}t. \tag{5.7}$$

Since the motion is irrotational to first order at least, $\partial w/\partial x$ may be replaced by $\partial u/\partial z$. Hence

$$\frac{\partial \zeta}{\partial x} = \int \frac{\partial u}{\partial z}\,\mathrm{d}t = \frac{\partial}{\partial z}\int u\,\mathrm{d}t = \frac{\partial}{\partial z}\,\Delta x. \tag{5.8}$$

Combining this result with (5.5) we find, correct to second order, that

$$\zeta_{\mathrm{L}} = \zeta_{\mathrm{E}} + \frac{\partial}{\partial z}\frac{1}{2}\,(\Delta x)^2, \tag{5.9}$$

where Δx is given by (5.4). This relates the vertical displacement ζ_{L} as measured by a free-floating buoy to that measured by a fixed probe.

The relation (5.9) can be applied in the first place to the narrowband model of §2. For, associated with the first-order terms $a\cos\theta$ there is a horizontal velocity

$$u = a\sigma\,\mathrm{e}^{kz}\cos\theta. \tag{5.10}$$

So on evaluating the second term in (5.9) at $z = 0$ we obtain

$$\zeta_{\mathrm{L}} = \zeta_{\mathrm{E}} + a^2k\sin^2\theta + O(a^3k^2). \tag{5.11}$$

From (2.1) this is

$$\zeta_{\mathrm{L}} = a\cos\theta + \tfrac{1}{2}a^2k, \tag{5.12}$$

correct to second order. In other words the motion is purely sinusoidal, apart from a term which varies only on the longer timescale of the wave groups. The latter represents a displaced mean level, midway between the level of crest and trough. The Eulerian mean level being taken as zero, it follows that this local mean level must be equal to the amplitude of the second harmonic in ζ_{E}, that is $\frac{1}{2}a^2k$; see figure 1.

Physically, the reason for this displaced mean is that a particle in the free surface lingers for longer near the wave crests, where it is moving forwards with the wave, than it does in the wave troughs, where its motion is opposite to the phase speed. Hence, Lagrangian averages will tend to overweight crest values and underweight trough values, relative to Eulerian averages. A first consequence is that the Lagrangian mean surface level is higher than the Eulerian.†

To calculate the moments μ_r of ζ_{L} from (5.12) we have, to lowest order,

$$\mu_1 = \tfrac{1}{2}\bar{a}^2k, \quad \mu_2 = \tfrac{1}{2}\bar{a}^2, \quad \mu_3 = \tfrac{3}{2}\bar{a}^4k. \tag{5.13}$$

Hence the cumulants are given by

$$\kappa_1 = \tfrac{1}{2}\bar{a}^2k, \quad \kappa_2 = \tfrac{1}{2}\bar{a}^2, \quad \kappa_3 = \tfrac{3}{4}\bar{a}^4k. \tag{5.14}$$

Remarkably, although the first cumulant κ_1 is now positive, the second and third cumulants are the same as for ζ_{E} (see (2.7)). Hence the coefficient of skewness $\lambda_3 = \kappa_3/\kappa_2^{\frac{3}{2}}$ is the same!

A qualitative explanation is as follows. In a wavetrain of *uniform* height the vertical displacement is indeed symmetric about its mean value; but that mean value is displaced from zero by a second-order amount depending on the wave steepness. Now even in a narrowband spectrum, the waves are not of uniform height. So the 'tails' of the distribution, which are due mainly to the larger waves, are shifted *more* in a positive sense, relative to the average, than is the region in the centre, which

† For uniform waves, this effect was noticed independently by I. D. James (personal communication).

depends partly on the lower waves. But the third moment of the distribution is influenced by the 'tails' more than is the mean value. The net effect is to produce a positive coefficient of skewness.

6. Lagrangian measurements in random wavefields

Equation (5.9) can easily be generalized to three dimensions to give

$$\boxed{\zeta_L = \zeta_E + \frac{\partial}{\partial z}\frac{1}{2}\left\{\left(\int u\,dt\right)^2 + \left(\int v\,dt\right)^2\right\},} \tag{6.1}$$

where v is the y-component of the particle velocity, and this may be used to evaluate the skewness in a random wavefield.

Adopting the approach of Longuet-Higgins (1963), in which the first-order motion is represented by

$$\zeta = \sum_{i=1}^{N} a_i \cos\theta, \quad \theta_i = \boldsymbol{k}_i\cdot\boldsymbol{x} - \sigma_i t + \epsilon_i, \tag{6.2}$$

the phases ϵ_i being random, we find

$$\frac{\partial}{\partial z}\frac{1}{2}\left\{\left(\int u\,dt\right)^2 + \left(\int v\,dt\right)^2\right\} = \frac{1}{2}\sum_{i,j} a_i a_j \frac{\boldsymbol{k}_i\cdot\boldsymbol{k}_j}{k_i k_j}(k_i + k_j)\sin\theta_i \sin\theta_j, \tag{6.3}$$

where $k_i = |\boldsymbol{k}_i|$. To obtain ζ_L we have only to add the above terms to the right-hand side of the (corrected) equation (3.7)* for $\zeta_E^{(2)}$. Following through the argument of that section we find that, to a second approximation,

$$\zeta_L = \sum_i \alpha_i \xi_i + \sum_{i,j} \alpha_{ij}\xi_i\xi_j, \tag{6.4}$$

where

$$\alpha_i = \begin{cases} 1 & (i = 1, 2, \ldots, N) \\ 0 & (i = (N+1), \ldots, 2N), \end{cases} \tag{6.5}$$

as before, but now

$$\alpha_{ij} = \begin{cases} \frac{1}{2}(k_i, k_j)^{\frac{1}{2}}\{B^-_{i,j} + B^+_{i,j} - \boldsymbol{k}_i\cdot\boldsymbol{k}_j + (k_i + k_j)(k_i k_j)^{-\frac{1}{2}}\}, & \text{when } i,j = 1, 2, \ldots, N, \\ \frac{1}{2}(k_i k_j)^{-\frac{1}{2}}\{B^-_{i,j} - B^+_{i,j} - \boldsymbol{k}_i\cdot\boldsymbol{k}_j + (k_i + k_j)(k_i k_j)^{-\frac{1}{2}} k_i k_j\} & \\ \qquad\qquad\qquad\qquad \text{when } i,j, = (N+1), \ldots, 2N, & \\ 0 \quad \text{otherwise.} & \end{cases} \tag{6.6}$$

In (6.4) the ξ_i denote independent random variables, $a_i\cos\theta_i$ or $-a_i\sin\theta_i$. The constants $B^+_{i,j}$ and $B^-_{i,j}$ are functions of $\boldsymbol{k}_i$ and $\boldsymbol{k}_j$ given by equations (3.8)*. When $i = j$, then $B^+_{i,j}$ and $B^-_{i,j}$ both vanish and we have

$$\alpha_{ii} = \tfrac{1}{2}k_i \quad (i = 1, 2, \ldots, 2N). \tag{6.7}$$

The expressions for the cumulants then become, in integral form,

$$\left.\begin{aligned} \kappa_1 &= \iint kE(\boldsymbol{k})\,d\boldsymbol{k}, \\ \kappa_2 &= \iint E(\boldsymbol{k})\,d\boldsymbol{k}, \\ \kappa_3 &= 6\iint K(\boldsymbol{k}, \boldsymbol{k}')\,E(\boldsymbol{k})\,E(\boldsymbol{k}')\,d\boldsymbol{k}\,d\boldsymbol{k}', \end{aligned}\right\} \tag{6.8}$$

where $E(\boldsymbol{k})$ denotes the two-dimensional spectral density and $K(\boldsymbol{k}, \boldsymbol{k}')$ is the same function as given in (3.12)*.

In the one-dimensional case these equations reduce to

$$\left.\begin{aligned}\kappa_1 &= \int_0^\infty kF(\sigma)\,\mathrm{d}\sigma \quad (k = \sigma^2/g),\\ \kappa_2 &= \int_0^\infty F(\sigma)\,\mathrm{d}\sigma,\\ \kappa_3 &= 3\int_0^\infty \int_0^\infty \min(k, k')\,F(\sigma)\,F(\sigma')\,\mathrm{d}\sigma\,\mathrm{d}\sigma'.\end{aligned}\right\} \tag{6.9}$$

The only difference between these expressions and those for the Eulerian cumulants, (2.9)–(2.11), lies in the value of κ_1. Whereas for the Eulerian cumulants $\kappa_1 = 0$ (which is a consequence of the choice of origin for z), in the Lagrangian case κ_1 is positive, on account of the second harmonic in ζ_{E}, which raises both crests and troughs by an equal, second-order quantity.

However, the non-zero value of κ_1 has no effect upon the values of κ_2 and κ_3. Hence the measured skewness is unaltered, just as in the narrowband case (§5).

7. Conclusions

We have shown by a simple model that in a narrowband, unidirectional sea the skewness λ_3 and the 'significant slope' s are related by (2.16), not (1.1), and that in a broader spectrum the ratio $\lambda_3/\pi s$ may have a rather wide range of values, as shown in table 3. This conclusion is consonant with the available field data (§4) and there may be reasons why laboratory measurements are not truly representative of ocean wave conditions.

We have derived a general relation (5.9) between the surface elevation ζ_{E} as measured in an Eulerian sense, say by a fixed probe, and the corresponding Lagrangian elevation ζ_{L} as recorded by an ideal small float. This relation is generalized in (6.1). When the statistical properties of ζ_{E} and ζ_{L} are compared, it is found, contrary to expectation, that the skewness and the variance in the two records are equal, although the apparent mean level in the Lagrangian record is slightly raised. Thus the relation between λ_3 and s is the same. The change in mean level, which would of course not be noticed by an accelerometer, is due to the fact that particles in the surface remain somewhat longer near the crests of the waves than in the troughs.

In practice, Lagrangian wave observations are often made by means of accelerometer buoys which have a response falling off at low frequencies. For our theoretical conclusions to apply to such measurements, it appears necessary that the frequency range should include at least the group frequencies. The possible effect of mooring forces on the buoy motions is left for a separate study.

Appendix. Derivation of equation (3.12)

On substituting $(\beta/\sigma)^m = \xi$, $(\beta/\sigma')^m = \eta$ in (2.12) and (3.9) we have

$$\kappa_3 = \frac{6\alpha^2}{m^2 g \beta^{2(n-2)}} \int_0^\infty \left\{\int_\eta^\infty \xi^{(n-3)/m-1}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi\right\} \eta^{(n-1)/m-1}\,\mathrm{e}^{-\eta}\,\mathrm{d}\eta. \tag{A 1}$$

This may be written

$$\kappa_3 = \frac{6\alpha^2}{m^2 g\beta^{2(n-2)}} \int_0^\infty \Gamma\left(\frac{n-3}{m}, \eta\right) \eta^{(n-1)/m-1} \, \mathrm{e}^{-\eta} \, \mathrm{d}\eta, \tag{A 2}$$

where $\Gamma(z,\eta)$ is the incomplete gamma function:

$$\Gamma(z,\eta) = \int_\eta^\infty \mathrm{e}^{-t} \, t^{z-1} \, \mathrm{d}t. \tag{A 3}$$

Equation (A 2) may be further simplified by using a result from Erdélyi *et al.* (1953, vol. II, p. 138), to obtain

$$\kappa_3 = \frac{6\alpha^2}{m^2 g\beta^{2(n-2)}} \, 2^{-\frac{3}{2}} \Gamma(\tfrac{3}{2}) \, {}_2F_1\left(1, \frac{2(n-2)}{m}; \frac{n-1}{m}+1, \tfrac{1}{2}\right), \tag{A 4}$$

where ${}_2F_1$ is a generalized hypergeometric function. On using the definitions of λ_3 in (2.8), κ_2 in (2.11) and s in (2.14), together with $\sigma_\mathrm{p} = \beta(m/n)^{1/m}$ and $\kappa_2 = \mu_2 = 0$, given by (3.10), we obtain equation (3.12).

In the special case $m = n-1$ (which for $n = 5$ gives the P–M spectrum) it is possible to reduce (3.12) by use of the relations

$${}_2F_1(p, 1-q; p+1; \eta) = p\eta^{-p} B_\eta(p,q), \tag{A 5}$$

where

$$B_\eta(p,q) = \int_0^\eta t^{p-1}(1-t)^{q-1} \, \mathrm{d}t \tag{A 6}$$

(Erdélyi *et al.* 1953, vol. I, p. 87). This leads to

$$\lambda_3 = 6\pi s \frac{n-1}{n-3}\left(\frac{n}{n-1}\right)^{2/(n-1)} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (2^{(n-3)/(n-1)} - 1). \tag{A 7}$$

However, as $n \to \infty$ (A 7) does not reduce to (3.12) owing to non-uniform convergence in the narrowband case. In the special case considered here, we have also from (3.11) that

$$\nu^2 = \Gamma\left(\frac{n-3}{n-1}\right) \Big/ \Gamma\left(\frac{n-2}{n-1}\right) - 1. \tag{A 8}$$

Some numerical values derived from (A 7) and (A 8) are shown in table 3. In this case the ratio $\lambda_3/\pi s$ does not differ greatly from 7. But in the more general case (table 2) the variation is considerably greater.

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n	ν	$\lambda_3/\pi s$
4	0.679	7.103
5	0.425	6.965
6	0.314	6.952
10	0.157	7.072
100	0.013	7.463
∞	0	7.520

TABLE 3. Parameters ν and $\lambda_3/\pi s$ for the spectrum (3.9), when $m = n-1$

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